Name (IN CAPITALS): **Version #1**

Instructor: Shaun The Sheep

Math 10560 Exam 2
Mar. 21, 2024.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
11.	
12.	
13.	
14.	
Total	

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2	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
11.	________
12.	________
13.	________
14.	________
Total	________

Multiple Choice

1.(6pts) Use Simpson's rule with n=4n = 4 to estimate

$$\int_1^3 \frac{1}{x}dx.$$

The Simpson's rule is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}(f(x_0) + 4f(x_1) + 2f(x_2) \cdots 4f(x_{n-1}) + f(x_n)).$$

In this problem, $n = 4$, $x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$, $f(x) = \frac{1}{x}$.

(a) $\frac{1}{6}\left[1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{1}{3}\right]$

(b) $\frac{1}{4}\left[1 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{1}{3}\right]$

(c) $\frac{1}{6}\left[1 + \frac{4}{3} + \frac{1}{2} + \frac{4}{5} + \frac{1}{3}\right]$

(d) $\frac{1}{4}\left[1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{1}{3}\right]$

(e) $\frac{1}{3}\left[2 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{2}{3}\right]$

2.(6pts) Evaluate the improper integral

$$I := \int_3^5 \frac{1}{(x-3)^{1/3}}dx.$$

The integrand is not defined at the lower bound $x = 3$. For $3 < b \leq 5$

$$\int_b^5 \frac{a}{(x-3)^{1/3}}dx = \frac{3}{2}(x-3)^{2/3}\Big|_b^5$$
$$= \frac{3}{2}[2^{2/3} - (b-3)^{2/3}]$$

Then writing I as an improper integral:

$$I = \int_3^5 \frac{1}{(x-3)^{1/3}}dx = \lim_{b\to 3^+} \int_b^5 \frac{a}{(x-3)^{1/3}}dx = \lim_{b\to 3^+} \frac{3}{2}[2^{2/3} - (b-3)^{2/3}] = \frac{3}{2}2^{2/3} = \frac{3}{2^{1/3}}$$

(a) $\frac{3}{2^{1/3}}$

(b) $2^{2/3}$

(c) The integral diverges

(d) $\frac{2^{5/3}}{3}$

(e) $\frac{3\left(2^{2/3} - 3^{2/3}\right)}{2}$

3.(6pts) Which of the following is the correct expression for the arc length of the curve

$$y = e^{\sqrt{x}}$$

between the points $(1, e)$ and $(2, e^{\sqrt{2}})$?

The arc length formula for $y = f(x), a < x < b$ is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Here $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$. So the arc length for the function and interval given is

$$\int_1^2 \sqrt{1 + \left(\frac{1}{2\sqrt{x}} e^{\sqrt{x}}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{1}{4x} e^{2\sqrt{x}}\right)} dx$$

(a) $\int_1^2 \sqrt{1 + \frac{e^{2\sqrt{x}}}{4x}} dx$

(b) $\int_1^2 \sqrt{1 + \frac{e^x}{4x}} dx$

(c) $\int_1^2 \sqrt{1 + e^x} dx$

(d) $\int_1^2 \sqrt{1 - \frac{e^{2\sqrt{x}}}{4x}} dx$

(e) $\int_1^2 \sqrt{1 - \frac{e^x}{4x}} dx$

4.(6pts) Use Euler's method with step size 0.5 to estimate $y(0)$ where $y(x)$ is the solution to the initial value problem

$$y' = 2x - y^2, \quad y(-1) = -2.$$

$y'(-1) = 2(-1) - (-2)^2 = -6,$
$y(-0.5) \approx y(-1) + 0.5y'(-1) = -2 + 0.5 \cdot (-6) = -5$
$y'(-0.5) \approx 2(-0.5) - (-5)^2 = -26,$
$y(0) \approx y(-0.5) + 0.5y'(-0.5) \approx -5 + 0.5 \cdot (-26) = -18.$

(a) -18 (b) -31 (c) -2 (d) $-\frac{3}{2}$ (e) 7

5.(6pts) Find the solution to the initial value problem:

$$\frac{dy}{dx} = y\sin(x), \quad y\left(\frac{\pi}{2}\right) = 2.$$

The differential equation is separable. So we have

$$\int \frac{1}{y}dy = \int \sin(x)dx \implies \ln(|y|) = -\cos(x) + C.$$

Then we take exponential function on both sides to get

$$y = Ke^{-\cos(x)},$$

(notice that K is positive due to our initial condition, i.e. $|y| = y$).

Finally, plug in the initial value to get $2 = Ke^{-\cos(\frac{\pi}{2})} = K$. Therefore

$$y = 2e^{-\cos(x)}$$

(a) $y = 2e^{-\cos(x)}$

(b) $y = \sqrt{4 - \cos(x)}$

(c) $y = 2e^{1-\cos(x)}$

(d) $y = \sqrt{4 + \cos(x)}$

(e) $y = 2 + e^{-\cos(x)}$

6.(6pts) Consider the following **sequences**:

$$(I) \quad \left\{(-1)^n \frac{n^2-1}{1-3n^2}\right\}_{n=1}^{\infty} \qquad (II) \quad \left\{\frac{(-1)^n \sin\left(\frac{1}{n}\right)}{\sin\left(\frac{1}{n}\right)+1}\right\}_{n=1}^{\infty} \qquad (III) \quad \left\{\frac{n!}{(n+2)!}\right\}_{n=1}^{\infty}$$

Which of the following statements is true?

(I) diverges: Note that $\lim_{n\to\infty}\left|(-1)^n \frac{n^2-1}{1-3n^2}\right| = \lim_{n\to\infty} \frac{n^2-1}{3n^2-1} = \lim_{x\to x} \frac{x^2-1}{3x^2-1} = -1/3 \neq 0.$

(II) converges : $\lim_{n\to\infty}\left|\frac{(-1)^n \sin\left(\frac{1}{n}\right)}{\sin\left(\frac{1}{n}\right)+1}\right| = \lim_{n\to\infty} \frac{\sin(1/n)}{\sin(1/n)+1} = \frac{\sin(0)}{\sin(0)+1} = 0.$

(III) Converges: By the definition for factorials: $\lim_{n\to\infty} \frac{n!}{(n+2)!} = \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{\infty} = 0.$

(a) Sequences II and III converge and sequence I diverges.

(b) Sequence II converges and sequences I and III diverge.

(c) Sequence III converges and sequences I and II diverge.

(d) All three sequences converge.

(e) All three sequences diverge.

7.(6pts) Find the sum of the following series

$$\sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{5^{n-1}}.$$

Note $->$ This series starts at $n = 2$.

We rearrange the sum to apply the formula for geometric series.

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{5^{n-1}} &= \frac{(-1)^2 3^2}{5} \sum_{n=2}^{\infty} \frac{(-1)^{n-2} 3^{n-2}}{5^{n-2}} \\
&= \frac{9}{5} \sum_{n=1}^{\infty} \left(\frac{-3}{5}\right)^{n-1} \\
&= \frac{9}{5} \cdot \frac{1}{1-(-3/5)} = \frac{9}{8}
\end{aligned}$$

(a) $\frac{9}{8}$

(b) $\frac{9}{2}$

(c) $\frac{5}{2}$

(d) $-\frac{9}{2}$

(e) This series diverges.

8.(6pts) Find the sum of the following series

$$\sum_{n=1}^{\infty}\left[\frac{2n}{n+1}-\frac{2n+2}{n+2}\right].$$

The partial sums are telescoping sums

$$\sum_{n=1}^{N}\left[\frac{2n}{n+1}-\frac{2n+2}{n+2}\right]=(\frac{2}{2}-\frac{4}{3})+(\frac{4}{3}-\frac{6}{4})+\cdots+(\frac{2N}{N+1}-\frac{2N+2}{N+2})=1-\frac{2N+2}{N+2}.$$

Then

$$\sum_{n=1}^{\infty}\left[\frac{2n}{n+1}-\frac{2n+2}{n+2}\right]=\lim_{N\to\infty}\sum_{n=1}^{N}\left[\frac{2n}{n+1}-\frac{2n+2}{n+2}\right]=\lim_{n\to\infty}1-\frac{2N+2}{N+2}=-1.$$

(a) -1 (b) 1 (c) $-\frac{7}{3}$ (d) $-\frac{1}{3}$ (e) 2

9.(6pts) Find the solution of the differential equation

$$\frac{dy}{dx} - \left[\frac{2x}{x^2+4}\right] y = x(x^2+4)$$

with initial condition $y(0) = 1$.

Integrating Factor with $p(x) = -\frac{2x}{x^2+4}$:

$$I(x) = \exp\left(\int p(x)dx\right) = \exp\left(\int -\frac{2x}{x^2+4}dx\right)$$

The substitution $u = x^2+4$, $du = 2x\ dx$ gives

$$\int -\frac{2x}{x^2+4}dx = -\int \frac{du}{u} = -\ln|u| = -\ln(x^2+4)$$

$I(x) = \exp(-\ln(x^2+4)) = \frac{1}{x^2+4}$. Then

$$y = \frac{1}{I(x)}\int I(x)q(x)dx$$

where $q(x) = x(x^2+4)$. Therefore

$$y = (x^2+4)\int \frac{x(x^2+4)}{x^2+4}dx = (x^2+4)\int xdx = (x^2+4)\left(\frac{x^2}{2}+C\right).$$

Check initial value:

$$1 = y(0) = (0^2+4)(0^2+C) = 4C,$$

i.e. $C = 1/4$.

$$y = (x^2+4)\left(\frac{x^2}{2}+\frac{1}{4}\right) = \frac{x^2(x^2+4)}{2} + \frac{x^2+4}{4}$$

(a) $y = \dfrac{x^2(x^2+4)}{2} + \dfrac{x^2+4}{4}$

(b) $y = \dfrac{(x^6/6)+2x^4+8x^2+4}{x^2+4}$

(c) $y = \dfrac{x^2(x^2+4)}{2} + 1$

(d) $y = \dfrac{x^4+2x^2+4}{2} - 1$

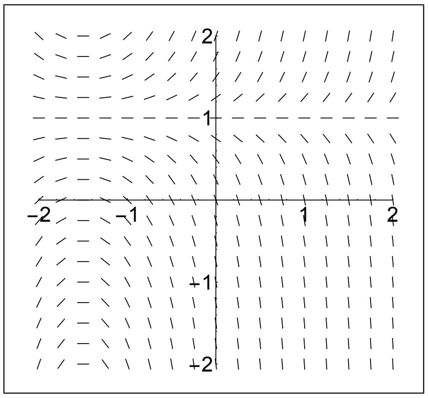
(e) $y = \dfrac{x^2+8}{2(x^2+4)}$

10.(6pts) Which of the following gives the direction field for the differential equation

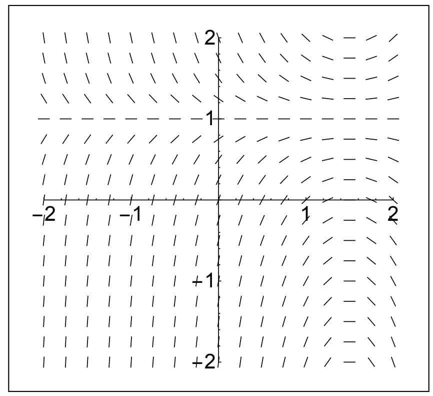
$$\frac{dy}{dx} = (2x+3)(y-1)\ ?$$

Notice that $\frac{dy}{dx} = 0$ on the lines $y = 1$ and $x = -3/2$. Also, $\frac{dy}{dx}$ is negative at the origin $(= -3)$. These conditions are only satisfied for option (a) below.

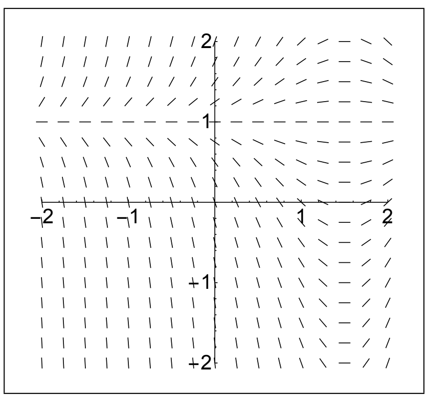
(a)



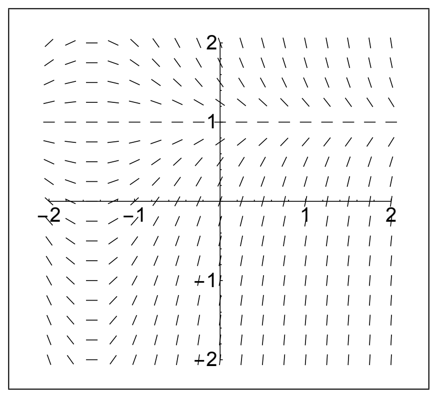
(b)



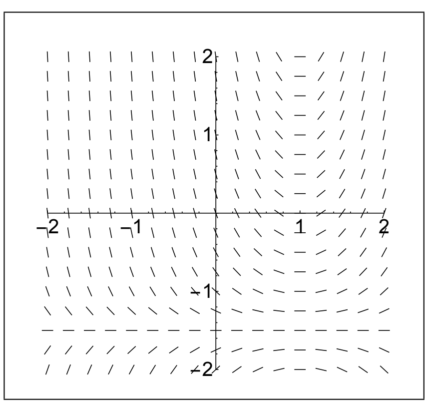
(c)



(d)



(e)



Partial Credit

Please justify all of your answers and show all of your work for credit on Question 11-13

11.(13pts) Evaluate the integral

$$I := \int_{4/\pi}^{\infty} \frac{\sin(1/x)}{x^2} dx.$$

Start by writing I as an improper integral:

$$I = \lim_{t \to \infty} \int_{4/\pi}^{t} \frac{\sin(1/x)}{x^2} dx.$$

Next use the substitution

$$u = 1/x, \quad du = -dx/x^2$$
$$u(4/\pi) = \pi/4, \quad u(t) = 1/t,$$

so that

$$I = \lim_{t \to \infty} \int_{\pi/4}^{1/t} -\sin(u) du = \lim_{t \to \infty} \int_{1/t}^{\pi/4} \sin(u) du.$$

Here we can take the limit, giving us a proper integral:

$$I = \int_0^{\pi/4} \sin(u) du = -\cos(u)\Big|_0^{\pi/4} = \cos(0) - \cos(\pi/4) = 1 - \frac{\sqrt{2}}{2}.$$

12.(13pts) Find the arc length of the curve $y = f(x)$ from the point $(0, 0)$ to the point $(\pi/3, -\ln(2))$, where

$$f(x) = \frac{\ln(\cos^2(x))}{2}.$$

Note $->$ The formula sheet will help with part of this calculation.

Recall the formula for the arclength ℓ of $f(x)$ from point $(a, f(a))$ to point $(b, f(b))$:

$$\ell = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

We compute $f'(x)$ with the chain rule:

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\cos^2(x)}(2\cos(x))(-\sin(x)) = -\tan(x)$$

Now, we find ℓ using a version of the *Pythagorean identity* (see formula sheet):

$$\ell = \int_0^{\pi/3} \sqrt{1 + (-\tan(x))^2} dx = \int_0^{\pi/3} \sqrt{1 + \tan^2(x)} dx = \int_0^{\pi/3} \sec(x) dx.$$

The antiderivative of $\sec(x)$ is also given on the formula sheet:

$$\ell = \ln|\sec(x) + \tan(x)| \Big|_0^{\pi/3} = \ln(\sec(\pi/3) + \tan(\pi/3)) - \ln(1 + 0) = \ln(2 + \sqrt{3})$$

13.(13pts) Find the orthogonal trajectories of the family of curves $y = Ke^{-2x}$.

The slopes of the family of curves at the point (x, y) are given by

$$m = \frac{dy}{dx} = -2Ke^{-2x} = -2y$$

Thus the curves intersecting our family orthogonally have slopes

$$m_{\perp} = -\frac{1}{m} = \frac{1}{2y}.$$

Solving the differential equation

$$\frac{dy}{dx} = \frac{1}{2y}$$

will give the family of orthogonal trajectories. Separating variables gives

$$2y\ dy = dx$$

and integrating both sides gives

$$y^2 = x + C$$

14.(1pts) You will be awarded this point if you write your name in CAPITALS and you mark your answers on the front page with an X (not an O) . You may also use this page for

ROUGH WORK

 Initials: ____________

The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x - y) - \cos(x + y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x - y) + \cos(x + y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$